SMOOTH FUNCTIONS WITH UNCOUNTABLY MANY ZEROS

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Abstract. In this short note we show that there exist uncountably generated algebras every non-zero element of which is a smooth function having uncountably many zeros. This result complements some recent ones by Enflo et al. [7, 9].

As it nowadays is common terminology, a subset M of a topological vector space X is called *lineable* (respectively, *spaceable*) in X if there exists an infinite dimensional linear space (respectively, infinite dimensional *closed* linear space) $Y \subset M \cup \{0\}$. Recently there have been several results regarding the linear structure of certain subsets of real functions having a *large* set of zeros. For instance, in [9], Enflo et al. proved that, for every infinite dimensional closed subspace X of $\mathcal{C}[0,1]$, the set of functions in X having infinitely many zeros in $[0,1]$ is *spaceable* in X. Also, in [7], Conejero et al. constructed an algebra of functions $\mathcal{A}$ enjoying the following properties: (i) $\mathcal{A}$ is uncountably infinitely generated (that is, the cardinality of a minimal system of generators of $\mathcal{A}$ is uncountable), (ii) every nonzero element of $\mathcal{A}$ is nowhere analytic, (iii) $\mathcal{A} \subset \mathcal{C}^\infty(\mathbb{R})$, (iv) every element of $\mathcal{A}$ has infinitely many zeros in $\mathbb{R}$, and (v) for every $f \in \mathcal{A} \setminus \{0\}$ and $n \in \mathbb{N}$, $f^{(n)}$ (the n-th derivative of f) enjoys the same properties as the elements in $\mathcal{A} \setminus \{0\}$. Also, let us recall the notion of *algebrability* (see, e.g. [1–5, 10]). Given an algebra $\mathcal{A}$, a subset $\mathcal{B} \subset \mathcal{A}$, and a cardinal number κ, we say that $\mathcal{B}$ is: (i) algebrable if there is a subalgebra $\mathcal{C}$ of $\mathcal{A}$ so that $\mathcal{C} \subset \mathcal{B} \cup \{0\}$ and the cardinality of any system of generators of $\mathcal{C}$ is infinite. (ii) κ-algebrable if there exists a κ-generated subalgebra $\mathcal{C}$ of $\mathcal{A}$ with $\mathcal{C} \subset \mathcal{B} \cup \{0\}$. (iii) strongly κ-algebrable if there exists a κ-generated free algebra $\mathcal{C}$ contained in $\mathcal{B} \cup \{0\}$.

On a totally different framework, and somehow related to the study of the set of zeros of functions on a given interval, Aron and Gurariy in 2003, asked whether there exists an infinite dimensional subspace of ℓ_∞ every non-zero element of which has a finite number of zeros. This question was recently answered, in the negative, in [6].

Let us also recall that both of the results from [7, 9] share a common ground: The cardinality of the considered set of zeros was always countable. Of course, by means of a Baire category argument (as seen in [8]) one can show that *almost every* continuous function having zeros has, actually, an uncountable amount of them.

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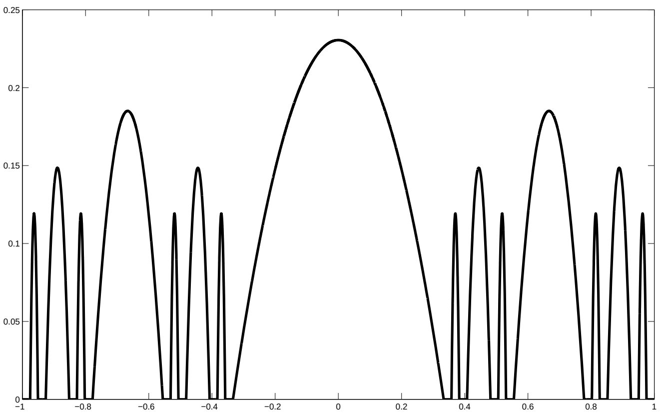

FIGURE 1. Sketch of $d(x)$ on $[-1,1]$.

In this short note we complement the previously mentioned results by proving, constructively, the following:

Theorem. *The subset of smooth functions in $\mathbb{R}$ having a uncountable set of zeros is strongly $\mathfrak{c}-$algebrable.*

Let us start by fixing $Z \subset \mathbb{R}$ with $|Z| = \mathfrak{c}$ and a function $0 \neq f \in \mathcal{C}^\infty(\mathbb{R})$ such that $f(z) = 0$ for every $z \in Z$ and f does not have horizontal asymptotes. Such a function can be defined as follows. Let $\mathfrak{C}$ be a copy of the Cantor set in the interval $[-1,1]$. Observe that $[-1,1] \setminus \mathfrak{C} = \bigcup_n I_n$, where the I_n's are pairwise disjoint open intervals. Now define the function $d : [-1,1] \to \mathbb{R}$ as

$$d(x) = \begin{cases} k_{a_n,b_n} \cdot (x-a_n)(b_n-x) & \text{if } x \notin \mathfrak{C}, \text{ and } x \in I_n = (a_n,b_n) \text{ for some } n, \\ 0 & \text{if } x \in \mathfrak{C}, \end{cases}$$

where k_{a_n,b_n} is a positive constant depending on a_n and b_n. Next, let g be the function, on $[-1,1]$, given by:

$$g(x) = \begin{cases} 0 & \text{if } x \in \mathfrak{C}, \\ e^{-1/d(x)} & \text{if } x \notin \mathfrak{C}. \end{cases}$$

The value of the constant k_{a_n,b_n} does not affect at all the smoothness of g. For instance, in Figures 1 and 2 we used $k_{a_n,b_n} = 1/(b_n-a_n)^{1.8}$. This constant plays the role of a "scaling factor".

We leave as an exercise to the reader to check that g is smooth. Next, we can define our function $f : \mathbb{R} \to \mathbb{R}$ by extending g in a usual way by making it smooth on $\mathbb{R}$ and by making it not have horizontal asymptotes.

Let us go back to our main construction now. Let $\mathcal{H}$ be a Hamel basis of $\mathbb{R}$ as a $\mathbb{Q}-$vector space such that all elements in $\mathcal{H}$ are positive. Also, let (for $r \in \mathcal{H}$ and $x \in \mathbb{R}$),

$$f_r(x) = e^{rx} \sin(f(x)).$$

Our aim is to show that the algebra generated by the f_r's, $A = \mathcal{A}(f_r : r \in \mathcal{H})$, is uncountably generated and that every element in A has an uncountable set of zeros.

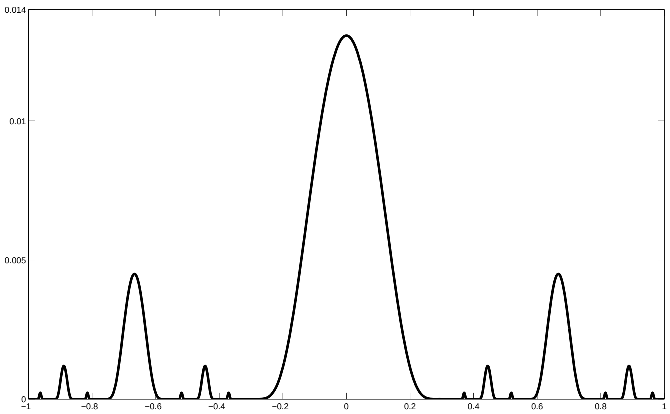

FIGURE 2. Sketch of $g(x)$ on $[-1,1]$.

In order to do so, let $k \in \mathbb{N}$, $P \in \mathbb{R}[z_1, z_2, \ldots, z_k]$ be any non-constant polynomial in k real variables, and $r_1, r_2, \ldots, r_k \in \mathcal{H}$. Now we need to see that:

(i.-) $\phi(z) := P(f_{r_1}, f_{r_2}, \ldots, f_{r_k})(z) = 0$ for every $z \in Z$.
(ii.-) The algebra A is $\mathfrak{c}-$generated.

First, notice that, since P can be written as

$$P(z_1, \ldots, z_k) = \sum_{j=1}^{q} a_j \cdot z_1^{n_{1,j}} \cdot \ldots \cdot z_k^{n_{k,j}},$$

with $q \in \mathbb{N}$, $\{n_{i,j} : 1 \leq i \leq k, 1 \leq j \leq q\} \subset \mathbb{N}$, and $a_j \in \mathbb{R} \setminus \{0\}$ for every $j \in \{1, \ldots, q\}$, then ϕ can be expressed as

$$\phi(z) = \sum_{j=1}^{q} a_j \cdot (\sin f(z))^{\sum_{i=1}^{k} n_{i,j}} \cdot e^{\sum_{i=1}^{k} (r_i n_{i,j} z)} = \sum_{j=1}^{q} a_j \cdot (\sin f(z))^{m_j} \cdot e^{z s_j},$$

where $m_j = \sum_{i=1}^{k} n_{i,j}$ and $s_j = \sum_{i=1}^{k} r_i n_{i,j}$ for $j \in \{1, \ldots, q\}$.

Once we have that, it is straightforward to check that $\phi(z) = 0$ for every $z \in Z$. Next, let us check some properties of the s_j's that appear in the expression of ϕ. First of all, notice that $s_j \neq 0$ for every $j \in \{1, \ldots, q\}$. Indeed, suppose that for some $j \in \{1, \ldots, q\}$ we have $s_j = 0$. Then, it would be

$$r_1 n_{1,j} + r_2 n_{2,j} + r_3 n_{3,j} + \cdots + r_k n_{k,j} = 0,$$

which contradicts the fact that $\mathcal{H}$ is a Hamel basis. Similarly it can be also shown that $s_i \neq s_j$ if $i \neq j$. Thus, we can assume without loss of generality, that $s_1 < s_2 < \cdots < s_q$.

Now, let us show that the set $\{f_r : r \in \mathcal{H}\}$ is algebraically independent. To achieve this, suppose that $\phi \equiv 0$, we shall show that $a_j = 0$ for every $j \in \{1, \ldots, q\}$. This will amount to $P \equiv 0$, and we will be done.

If $\phi \equiv 0$, then we would have that

$$\frac{\phi(z)}{e^{s_1 z}} = a_1 (\sin z)^{m_1} + \sum_{j=2}^{q} a_j \cdot (\sin z)^{m_j} \cdot e^{z(s_j - s_1)}$$

is also 0 for every $z \in \mathbb{R}$.

Let, now, take the limit when $z \to -\infty$. Then, we have that

$$\begin{aligned} 0 = & \lim_{z\to-\infty} a_1 (\sin f(z))^{m_1} + \sum_{j=2}^{q} a_j \cdot \lim_{z\to-\infty} (\sin f(z))^{m_j} \cdot e^{z(s_j-s_1)} \\ = & \ a_1 \cdot \lim_{z\to-\infty} (\sin f(z))^{m_1} + \sum_{j=2}^{q} a_j \cdot \lim_{z\to-\infty} \sin(f(z))^{m_j} e^{z(s_j-s_1)} = \\ = & \ a_1 \cdot \sin\left(\lim_{z\to-\infty} f(z)\right)^{m_j} + 0, \end{aligned}$$

and we obtain that $a_1 = 0$ (since f has no horizontal asymptotes). We can now proceed similarly (dividing now the expression $\sum_{j=2}^{q} a_j \cdot (\sin z)^{m_j} \cdot e^{s_j z}$ by $e^{s_2 z}$ and taking again limits when $z \to -\infty$) and we would obtain that all the a_j's are 0. Thus, $P \equiv 0$, the set $\{f_r : r \in \mathcal{H}\}$ is algebraically independent, and we are done.

Remark. Notice that this result is the best possible in terms of dimension, since the set of continuous functions has dimension $\mathfrak{c}$. Let us also recall that this construction can also be done using any other types of fractal sets with arbitrary fractal dimension. We chose the Cantor set for convenience.

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